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An Existence Theorem for a Class of Inclusions

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Abstract—In this paper, we obtain a general existence theorem for a class of inclusions, which extends the result obtained in [1].

Keywords—Variational inequalities, Multifunctions, Fixed points, Existence of zeros.

1. INTRODUCTION

In 1985, Ricceri [1] established a general existence theorem for variational inequalities associated with discontinuous multivalued operator that do not satisfied any type of monotonicity.

In a finite-dimensional setting, Theorem 1 of [1] was extended to quasi-variational inequalities by Cubiotti [2]. The full extension of Theorem 1 of [1] to quasi-variational inequalities was conjectured by Ricceri himself in [3]. Very recently, a partial, but significant, case of such conjecture was proved by Cubiotti and Yen in [4].

The aim of this paper is to establish a general theorem for a class of inclusions which, inspired by the above recalled Ricceri's result, admits this latter as a direct consequence. Precisely, we deal here with the following problem: given a real Hausdorff topological vector space E, a convex set $X \subseteq E$, and a multifunction $F : X \to 2^E$, to prove, under suitable assumptions, the existence of a point $x^* \in X$ such that $X \subseteq F(x^*)$.

In the final part of the paper, we present some consequences of Theorem 3.1 in the setting of reflexive, separable, Banach spaces. In particular, we prove an existence theorem for the zeros of nonlinear operators defined on a reflexive separable Banach space with range in a normed space.

2. PRELIMINARIES AND NOTATIONS

Given two topological spaces S and V, a multifunction $F : S \to 2^V$ is said to be lower semicontinuous in S if, for each open $\Omega \subseteq V$, the set $F^-(\Omega) = \{s \in S : F(s) \cap \Omega \neq \emptyset\}$ is open in S. The multifunction F is said to be upper semicontinuous in S if, for each closed $\Omega \subseteq V$, the set $F^-(\Omega)$ is closed in S. The multifunction F will be called continuous provided it is both lower and upper semicontinuous. The graph of F is the set $\{(s, v) \in S \times V : v \in F(s)\}$.

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Let E a real Hausdorff topological vector space, for $A \subseteq E$ we denote by $\text{aff}(A)$ the affine hull of the set A. Namely, we put

$$\text{aff}(A) = \left\{ \sum_{i=1}^{n} \lambda_i x_i : n \in \mathbb{N}, x_i \in A, \lambda_i \in \mathbb{R} \text{ and } \sum_{i=1}^{n} \lambda_i = 1 \right\}.$$

If $A \subseteq B \subseteq E$, we denote by $\text{int}_B(A)$ the interior of A in B, with $\text{int}(A)$ standing for $\text{int}_E(A)$. The relative interior of A, denoted by $\text{ri}(A)$, is the interior of A in $\text{aff}(A)$. That is, we put

$$\text{ri}(A) = \text{int}_{\text{aff}(A)}(A).$$

We recall that each nonempty finite-dimensional convex set $A \subseteq E$ has nonempty relative interior. Moreover, we denote by $\text{conv}(A)$ the convex hull of A; that is, we put

$$\text{conv}(A) = \left\{ \sum_{i=1}^{n} \lambda_i x_i : n \in \mathbb{N}, x_i \in A, \lambda_i \in [0,1] \text{ and } \sum_{i=1}^{n} \lambda_i = 1 \right\}.$$

The set $A \subseteq E$ is said to be co-convex if its complementary set is convex.

The set $A \subseteq E$ is said to be finitely closed if, for every finite-dimensional linear subspace S of E, the set $A \cap S$ is closed with respect to the usual Euclidean topology of S.

3. MAIN RESULT

Our main result is the following.

THEOREM 1. *Let E be a real Hausdorff topological vector space; V a linear subspace of E; X a convex subset of E, with $\text{ri}(X) \neq \emptyset$; K a compact subset of X; F a multifunction from X into E. Further, let $\mathcal{F}$ be a directed (by inclusion) family of finite-dimensional linear subspaces of V meeting K, with $V = \cup_{S \in \mathcal{F}} S$ and satisfying the following conditions:*

- (a_1) *for every $S \in \mathcal{F}$ and every compact convex set Y, with $K \cap S \subseteq Y \subseteq X \cap S$ and $\dim(Y) = \dim(X \cap S)$, one has $\text{ri}(Y) \backslash F(x) \neq \emptyset$ for all $x \in Y \backslash K$ and $x \notin \text{conv}(\text{ri}(Y) \backslash F(x))$ for all $x \in Y$;*
- (a_2) *for every $S \in \mathcal{F}$ and for every $y \in (X - X) \cap S$, the set*

$$\{x \in X \cap S : y \in x - F(x)\}$$

  is closed in $X \cap S$;
- (a_3) *for each $x \in X \cap \overline{V}$ such that $(\text{ri}(X) \cap V) \backslash F(x) \neq \emptyset$, there exist $y_0 \in \text{ri}(X)$, with $x - y_0 \in V$ and a neighborhood U of x such that $z - x + y_0 \notin F(z)$ for all $z \in U \cap K \cap V$.*

Then, there exists $x^* \in K$ such that $\text{ri}(X) \cap V \subseteq F(x^*)$.

PROOF. Fix $S \in \mathcal{F}$ and denote by L_S the affine hull of $X \cap S$. First, we will show that there exists a point $x_S \in K \cap S$ such that $\text{ri}(X \cap S) \subseteq F(x_S)$.

Since $\dim(L_S) < \infty$ and $X \cap S$ is convex, we have $\text{ri}(X \cap S) \neq \emptyset$. With respect to any metric on L_S inducing the topology of L_S, let $\{B_n\}$ be a sequence of open balls such that

$$\text{ri}(X \cap S) = \bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} \overline{B_n}.$$

For each $n \in \mathbb{N}$, put

$$Y_n = \text{conv}\left(\bigcup_{i=1}^{n} \overline{B_i} \cup (K \cap S) \right).$$

Of course, the sequence $\{Y_n\}$ is nondecreasing, each Y_n is a compact convex subset of $X \cap S$ containing $K \cap S$, with $\dim(Y_n) = \dim(X \cap S)$, and

$$\mathrm{ri}(X \cap S) = \bigcup_{n=1}^{\infty} \mathrm{ri}(Y_n).$$

Now, fix $n \in \mathbb{N}$. For each $x \in Y_n$, put

$$\Phi(x) = \mathrm{ri}(Y_n) \backslash F(x).$$

We claim that there is some $x \in Y_n$ such that $\Phi(x) = \emptyset$. Assume the contrary. Let us prove that the multifunction Φ is lower semicontinuous. To this end, let Ω be an open set in E and let $x_0 \in Y_n$, $y_0 \in \Phi(x_0) \cap \Omega$. Put

$$U_0 = \{x \in Y_n : x_0 - y_0 \notin x - F(x)\}.$$

By (a_2), U_0 is a neighborhood of x_0 in Y_n. Also, put

$$W_0 = U_0 \cap [(x_0 - y_0) + (\mathrm{ri}(Y_n) \cap \Omega)] \cap Y_n.$$

Observe that the set $[(x_0 - y_0) + (\mathrm{ri}(Y_n) \cap \Omega)]$ is (contained and) open in L_S and contains x_0. Therefore, W_0 is a neighborhood of x_0 in Y_n.

Let $x \in W_0$. Put $z = x - x_0 + y_0$. Then, $z \in \mathrm{ri}(Y_n) \cap \Omega$, and since $x \in U_0$, one has $x - z \notin x - F(x)$, that is $z \notin F(x)$. Hence, $z \in \Phi(x) \cap \Omega$ which proves the lower semicontinuity of Φ. Then, $x \to \mathrm{conv}(\Phi(x))(x \in Y_n)$ is a lower semicontinuous multifunction, with nonempty, convex, finite-dimensional values, and so, by Theorem 3.1‴ of [5], it admits a continuous selection. That is, there is a continuous function $f : Y_n \to Y_n$ such that $f(x) \in \mathrm{conv}(\Phi(x))$ for all $x \in Y_n$. By Brouwer's theorem, such f would admit a fixed point, say $\overline{x}$. Thus, $\overline{x} \in \mathrm{conv}(\mathrm{ri}(Y_n) \backslash F(\overline{x}))$, against (a_1). Consequently, there is some $x_n \in Y_n$ such that $\mathrm{ri}(Y_n) \subseteq F(x_n)$. By (a_1) again, we have $x_n \in K \cap S$. Consider the sequence $\{x_n\}$. Since $K \cap S$ is sequentially compact, there is a subsequence, still denoted by $\{x_n\}$, converging to a point $x_S \in K \cap S$. We claim that $\mathrm{ri}(X \cap S) \subseteq F(x_S)$. Assume the contrary. Let $\tilde{y} \in \mathrm{ri}(X \cap S) \backslash F(x_S)$. Let $\nu \in \mathbb{N}$ be such that $\tilde{y} \in \mathrm{ri}(Y_\nu)$. By (a_2), there is a neighborhood U_1 of x_S such that $x_S - \tilde{y} \notin x - F(x)$ for all $x \in U_1 \cap X \cap S$. Put

$$W_1 = U_1 \cap (x_S - \tilde{y} + \mathrm{ri}(Y_\nu)).$$

Since W_1 is a neighborhood of x_S in L_S, there is $n > \nu$ such that $x_n \in W_1$. If we put $y_n = x_n - x_S + \tilde{y}$, we have $y_n \in \mathrm{ri}(Y_\nu)$, and hence, $y_n \in F(x_n)$. From this, it clearly follows that $x_S - \tilde{y} \in x_n - F(x_n)$, against the fact that $x_n \in U_1 \cap X \cap S$.

Finally, since the family $\mathcal{F}$ is directed respect to inclusion, we can consider the generalized subsequence $\{x_S\}_{S \in \mathcal{F}} \subseteq K$. As K is compact, there is $\hat{x} \in K$, cluster point of $\{x_S\}_{S \in \mathcal{F}}$. We will show that $\mathrm{ri}(X) \cap V \subseteq F(\hat{x})$. Assume, to the contrary, that there is some $\hat{y} \in \mathrm{ri}(X) \cap V$ such that $\hat{y} \notin F(\hat{x})$. By (a_3), there exist a point $y_0 \in \mathrm{ri}(X)$ with $\hat{x} - y_0 \in V$ and a neighborhood U of $\hat{x}$ such that $z - \hat{x} + y_0 \notin F(z)$ for all $z \in U \cap K \cap V$.

Let

$$W = U \cap [(\hat{x} - y_0) + \mathrm{ri}(X)].$$

Taking into account that W is a neighborhood of $\hat{x}$ in $\mathrm{aff}(X)$, there exists $S_1 \in \mathcal{F}$ such that $\hat{x} - y_0 \in S_1$ and $x_{S_1} \in W$. If we put $y_{S_1} = x_{S_1} - \hat{x} + y_0$, we have $y_{S_1} \in F(x_{S_1})$ against the fact that $x_{S_1} \in U \cap K \cap V$. The proof is complete. ∎

REMARK 3.1. Theorem 1 of [1] can be obtained as a special case of our Theorem 3.1, taking (with the notations of [1])

$$F(x) = \left\{ y \in V : \inf_{\varphi \in \Phi(x)} \langle \varphi, x - y \rangle \leq 0 \right\}.$$

By Theorem 3.1 we can prove the following result.

THEOREM 3.2. *Let E be a space provided with two vector topologies τ_1, τ_2 with τ_1 weaker than τ_2. Let (E, τ_1) be a real Hausdorff topological vector space, V a linear dense subspace of (E, τ_2), X a convex subset of E with nonempty interior, respect to τ_1. Let K, K_1 be two subsets of X such that K is τ_1-compact and $K_1 \subseteq K$ is finite-dimensional. Moreover, let F be a multifunction from X to E such that $F(x)$ is co-convex for each $x \in X \cap V$, τ_2-closed for each $x \in K$, and satisfying the following conditions:*

- (b_1) *for every $y \in (X - X) \cap V$, the set $\{x \in X \cap V : y \in x - F(x)\}$ is finitely closed in $X \cap V$;*
- (b_2) *for every $y \in (X - X) \cap V$, the set $\{x \in K : y \in x - F(x)\}$ is τ_1-closed;*
- (b_3) *$x \in F(x)$, for all $x \in X \cap V$;*
- (b_4) *$K_1 \backslash F(x) \neq \emptyset$, for all $x \in (X \cap V) \backslash K$.*

Then there exists $x^* \in K$ such that $\overline{X} \subseteq F(x^*)$.

PROOF. Let $\mathcal{F}$ be a directed (by inclusion) family of finite dimensional linear subspaces of E containing K_1 and such that $V = \cup_{S \in \mathcal{F}} S$. Cleary, $\mathcal{F}$ is nonempty and every $S \in \mathcal{F}$ meets K. So, fixed $S \in \mathcal{F}$, let Y be a convex and τ_1-compact set with $K \cap S \subseteq Y \subseteq X \cap S$ and $\dim(Y) = \dim(X \cap S)$.

By (b_4), for all $x \in Y \backslash K$ there exists a point $y \in K_1 \backslash F(x) \subseteq Y \backslash F(x)$. Since $F(x)$ is closed with respect to τ_2 topology, $y \in \mathrm{ri}(Y) \backslash F(x)$. Furthermore, by (b_3), for all $x \in Y$, $x \notin \mathrm{ri}(Y) \backslash F(x) = \mathrm{conv}(\mathrm{ri}(Y) \backslash F(x))$.

Let $x \in K \cap \overline{V} = K$ be such that $(\mathrm{int}(X) \cap V) \backslash F(x) \neq \emptyset$. Since the set $x - (\mathrm{int}(X) \backslash F(x))$ is τ_2-open, there exists $z_0 \in x - (\mathrm{int}(X) \backslash F(x)) \cap V$. Thus, $z_0 = x - y_0 \in V$, with $y_0 \in \mathrm{int}(X) \backslash F(x)$. Also, by (b_2), the set $\{z \in K : x - y_0 \notin z - F(z)\}$ is a τ_1-neighborhood of x in K, consequently, we have $z - x + y_0 \notin F(z)$ for all $z \in U \cap K \cap V$. By Theorem 3.1, there exists $x^* \in K$ such that $\mathrm{ri}(X) \cap V \subseteq F(x^*)$. Since $F(x)$ is τ_2-closed, this implies that $\overline{X} \subseteq F(x^*)$, as desired. ∎

Now, as consequences of Theorem 3.2, we point out the following results.

THEOREM 3.3. *Let E be a reflexive separable Banach space, V a linear dense subspace of E, $X \subseteq E$ closed, convex with nonempty weak interior. Further, let $F : X \to 2^E$ be a multifunction such that $F(x)$ is closed for each $x \in X$, co-convex for each $x \in X \cap V$, and satisfying the following conditions:*

- (α_1) *for every $y \in (X - X) \cap V$, $F^-(y)$ is sequentially weakly closed in X;*
- (α_2) *$\theta_E \in F(x)$ for all $x \in X \cap V$.*

Then, at least one of the following two assertions does hold:

1. *there exists some $x^* \in X$ such that $X \subseteq x^* - F(x^*)$;*
2. *for every bounded $C \subseteq X$ with $\dim(C) < \infty$, the set $\{x \in X \cap V : C \subseteq x - F(x)\}$ is unbounded.*

PROOF. Assume that (2) does not hold. Let $r > 0$ such that $C \subseteq \overline{B}(0, r)$. We can apply Theorem 3.2 to the multifunction $G(x) = x - F(x)$, taking: $K = \overline{B}(0, r) \cap X$ and $K_1 = C$. ∎

THEOREM 3.4. *Let E be a reflexive separable Banach space, V a linear dense subspace of E, $X \subseteq E$ closed, convex with nonempty weak interior, Y a topological vector space. Let $G : X \to 2^Y$ be a multifunction with closed values, such that $G(x)$ is co-convex for all $x \in X \cap V$ $G(x)$, and $\theta_Y \in G(x)$. Further, let $\varphi : E \to Y$ a linear, continuous operator such that $G^-(y)$ is sequentially strongly closed in X for every $y \in \varphi(X \cap V)$. Then, at least one of the following two assertions does hold:*

1. *there exists $x^* \in X$ such that $\varphi(x^*) - \varphi(X) \subseteq G(x^*)$;*
2. *for every bounded $C \subseteq X$ with $\dim(C) < \infty$, the set $\{x \in X \cap V : \varphi(X) - \varphi(C) \subseteq G(x)\}$ is unbounded.*

PROOF. Apply Theorem 3.3 by taking

$$F(x) = \{v \in E : \varphi(v) \in G(x)\}. \qquad \blacksquare$$

THEOREM 3.5. *Let E be a reflexive separable Banach space, $X \subseteq E$ closed, convex with nonempty weak interior, Y a normed space. Let $A : X \to Y$ an operator which is sequentially continuous from the weak topology of E to the strong topology of Y. Then, for every linear continuous operator $\varphi : E \to Y$, at least one of the following two assertions does hold:*

(1) *there exists some $x^* \in X$ such that $\|\varphi(x^*) - \varphi(y) - A(x^*)\|_Y \geq \|A(x^*)\|_Y$ for all $y \in X$;*
(2) *for every vector dense subspace V of E and for every bounded $C \subseteq X$ with $\dim(C) < \infty$, the set $\{x \in X \cap V : \|\varphi(x) - \varphi(y) - A(x)\|_Y \geq \|A(x)\|_Y \ \forall y \in C\}$ is unbounded.*

PROOF. Apply Theorem 3.4 to the multifunction

$$G(x) = \{w \in Y : \|w - A(x)\|_Y \geq \|A(x)\|_Y\}. \qquad \blacksquare$$

In particular, by Theorem 3.5 we can get the following result for the existence of zeros of nonlinear operators defined on a reflexive separable Banach space with range in a normed space.

THEOREM 3.6. *Let E be a reflexive separable Banach space, Y a normed space, $\varphi : E \to Y$ linear, continuous, subjective operator, $A : E \to Y$ an operator which is sequentially continuous from the weak topology of E to the strong topology of Y. Moreover, assume that there exist a dense subspace V of E and $r > 0$ such that $\|\varphi(x) - A(x)\|_Y < \|A(x)\|_Y$, for all $x \in V$, with $\|x\|_E > r$. Then there exists $x^* \in E$ such that $A(x^*) = \theta_Y$.*

PROOF. It is sufficient to apply Theorem 3.5 with $X = E$ and $C = \{\theta_E\}$. $\blacksquare$

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